



9.2 The trace of a matrix

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Introduction

- In this topic, we will
 - Define the trace of a matrix
 - Look at some consequences
 - Consider an application





The trace

- Given $A: \mathbf{R}^n \rightarrow \mathbf{R}^n$, then linear operator
 - That is, $A \in \mathcal{L}(\mathbf{R}^n)$

we will define the *trace* of the matrix as
the sum of the diagonal entries

$$\text{tr}(A) = \sum_{k=1}^n a_{k,k}$$

- The *trace* is a function mapping $\text{tr} : \mathcal{L}(\mathbf{R}^n) \rightarrow \mathbf{R}$.





The trace is linear

Theorem

The *trace* is a linear function mapping $\text{tr} : \mathcal{L}(\mathbf{R}^n) \rightarrow \mathbf{R}$.

Proof:

If $A, B \in \mathcal{L}(\mathbf{R}^n)$, then the diagonal entries of $\alpha A + \beta B$ are

$$\alpha a_{k,k} + \beta b_{k,k}$$

$$\begin{aligned}\text{Thus, } \text{tr}(\alpha A + \beta B) &= \sum_{k=1}^n (\alpha a_{k,k} + \beta b_{k,k}) \\ &= \alpha \sum_{k=1}^n a_{k,k} + \beta \sum_{k=1}^n b_{k,k} \\ &= \alpha \text{tr}(A) + \beta \text{tr}(B) \quad \blacksquare\end{aligned}$$





The trace is linear

Theorem

The subset of all linear operators with zero trace is a subspace.

Proof:

If $A, B \in \mathcal{L}(\mathbf{R}^n)$ and have zero trace,

$$\begin{aligned}\operatorname{tr}(\alpha A + \beta B) &= \alpha \operatorname{tr}(A) + \beta \operatorname{tr}(B) \\ &= \alpha \cdot 0 + \beta \cdot 0 = 0\end{aligned}$$

Thus, if A and B have zero trace, so does $\alpha A + \beta B$. ■





Application of the trace

- From calculus, you may recall that for sufficiently small ε ,

$$x(t_0 + \varepsilon) \approx x(t_0) + \varepsilon \dot{x}(t_0)$$

where $\dot{x}(t)$ is the time derivative of $x(t)$

- For example, if $x(t) = e^{-t} \cos(t)$, then

$$\dot{x}(t) = -e^{-t} \cos(t) - e^{-t} \sin(t)$$

- Thus, $x(\varepsilon) = x(0 + \varepsilon)$

$$\approx x(0) + \varepsilon \dot{x}(0)$$

$$= 1 - \varepsilon$$

- Thus, $x(0.1) \approx 0.9$, which is quite close, as

$$x(0.1) = 0.900317000$$





Application of the trace

- One application of the trace is as follows:

$$\det(I + \varepsilon A) \approx 1 + \varepsilon \operatorname{tr}(A)$$

Proof:

We will only show this for an upper-triangular matrix U .

The diagonal entries of $I + \varepsilon U$ are $1 + \varepsilon u_{k,k}$

Because U is upper triangular, it follows that so is $I + \varepsilon U$.

Thus,

$$\begin{aligned} \det(I + \varepsilon U) &= (1 + \varepsilon u_{1,1})(1 + \varepsilon u_{2,2}) \cdots (1 + \varepsilon u_{n,n}) \\ &= 1 + \varepsilon u_{1,1} + \varepsilon u_{2,2} + \cdots + \varepsilon u_{n,n} + \varepsilon^2 (\text{lower order terms}) \\ &\approx 1 + \varepsilon (u_{1,1} + u_{2,2} + \cdots + u_{n,n}) \\ &\approx 1 + \varepsilon \operatorname{tr}(U) \blacksquare \end{aligned}$$





Application of the trace

- For example, if

$$A = \begin{pmatrix} 0.92330 & 0.71603 & 0.81771 \\ 0.69398 & 0.75781 & 0.91228 \\ 0.30259 & 0.64758 & 0.83417 \end{pmatrix}$$

then $\det(I + 0.1A) \approx 1 + 0.1 \cdot 2.51528 = 1.251528$

– In reality, $\det(I + 0.1A) \approx 1.259198$





Summary

- Following this topic, you now
 - Know the definition of the trace
 - Understand that it is a linear function
 - Are aware of an application thereof





References

- [1] [https://en.wikipedia.org/wiki/Trace_\(linear_algebra\)](https://en.wikipedia.org/wiki/Trace_(linear_algebra))





Acknowledgments

None so far.





Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using Consolas. Mathematical equations are prepared in MathType by Design Science, Inc. Examples may be formulated and checked using Maple by Maplesoft, Inc.

The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see

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